What do global PŁ functions look like?

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with
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PŁ condition

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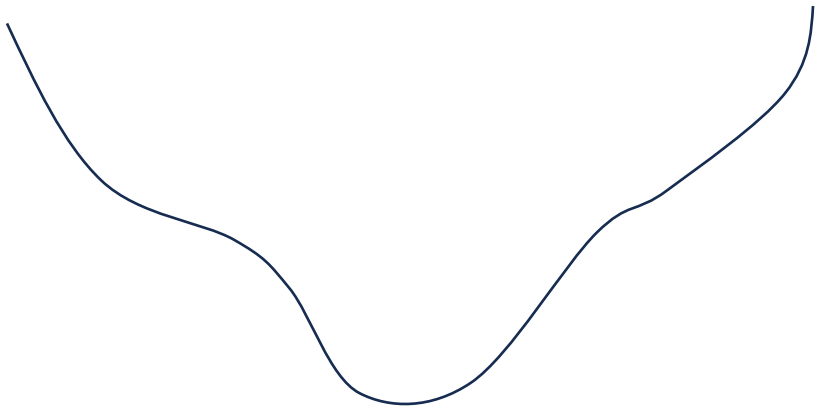
All critical points are global mins $\nabla f(x) = 0 \implies f(x) = f^*$



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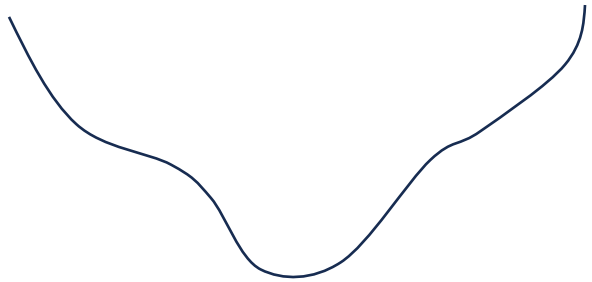

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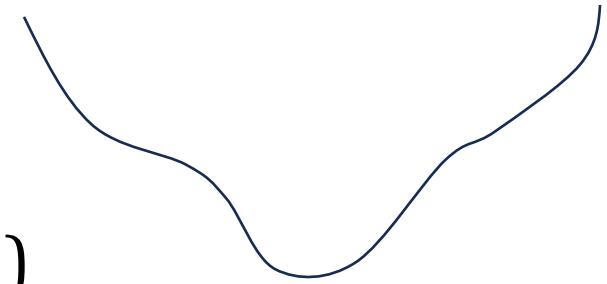

Gradient descent/flow behaves nicely

L-smooth and μ-PŁ imply linear convergence (short proof!)

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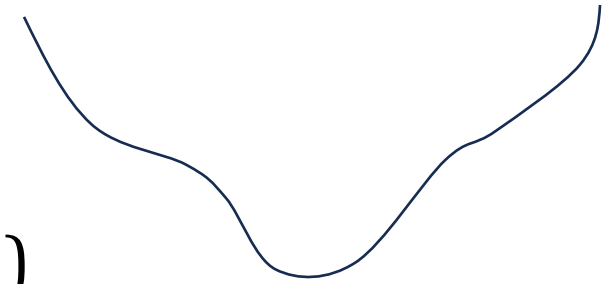

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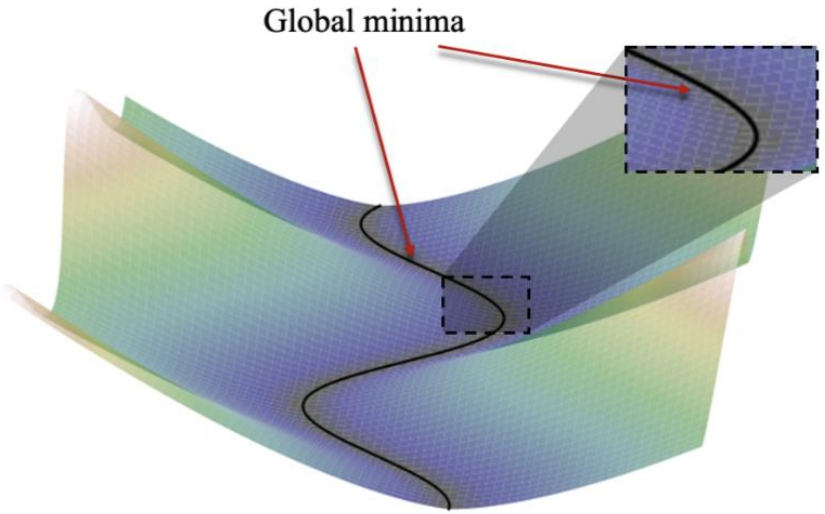

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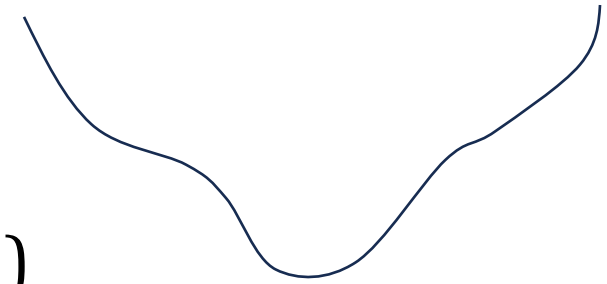

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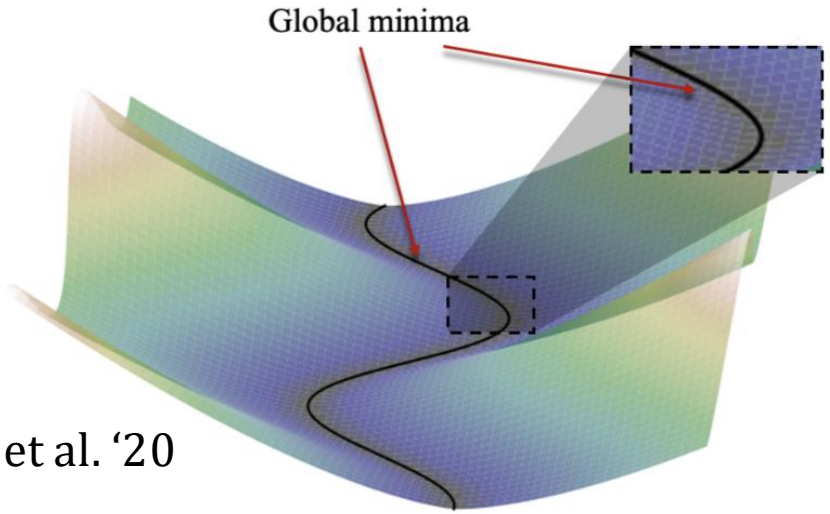
Overparameterization!

Applications: Deep learning (Liu et al. '20),

control (Fazel et al. '18), etc.



Liu et al. '20

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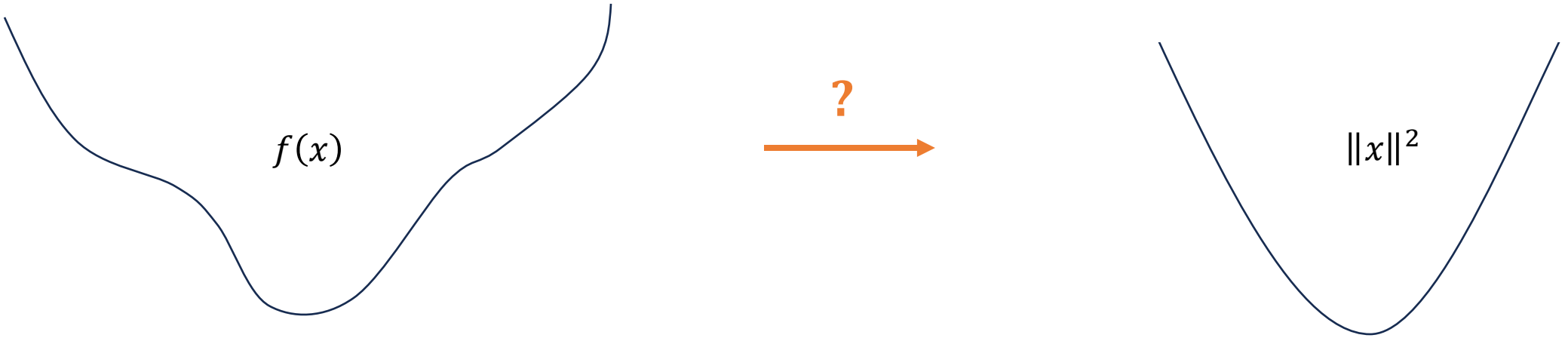

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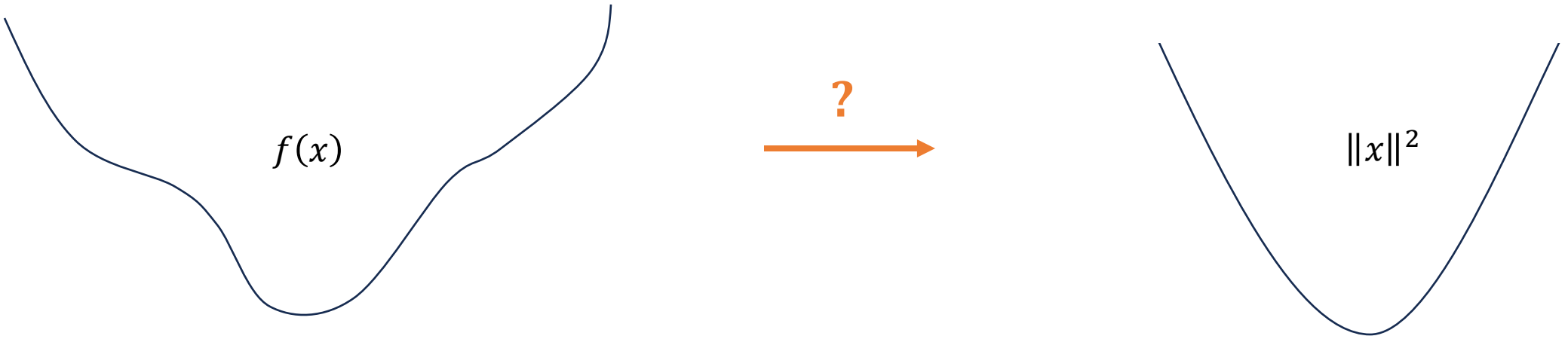

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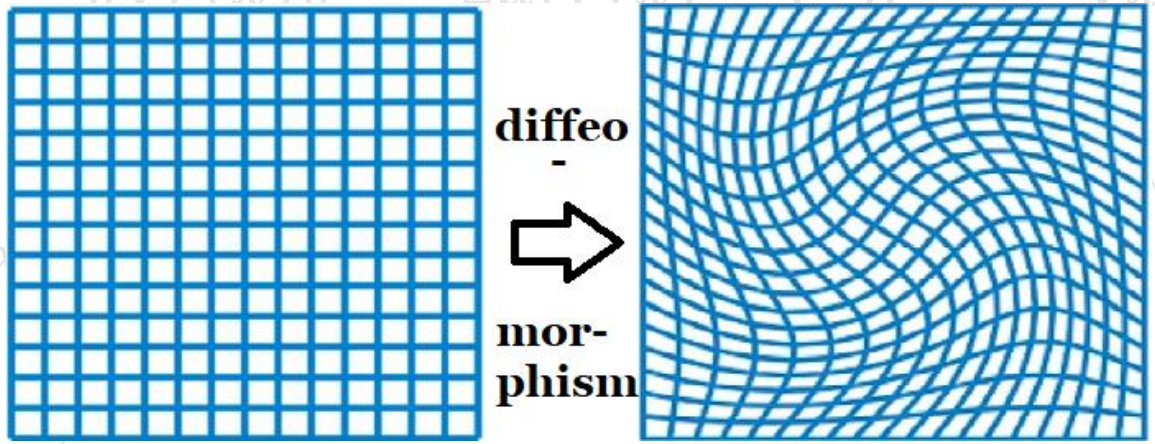

A diffeomorphism is a smooth map $\mathbb{R}^n \to \mathbb{R}^n$ with smooth inverse

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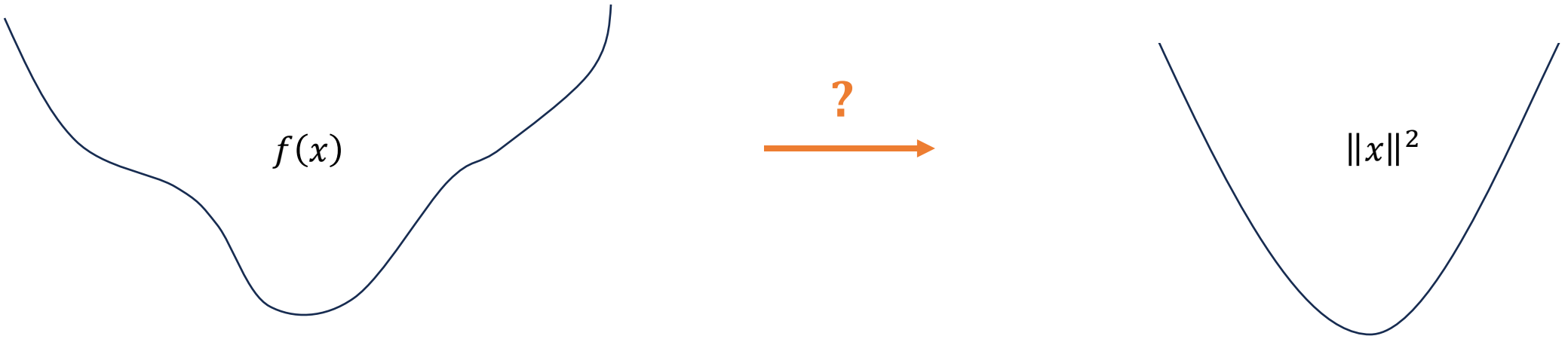

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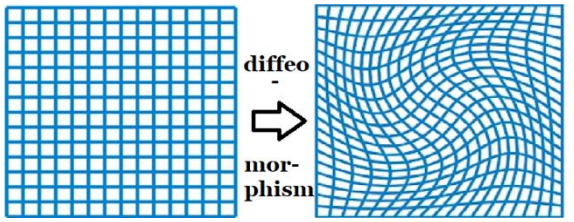

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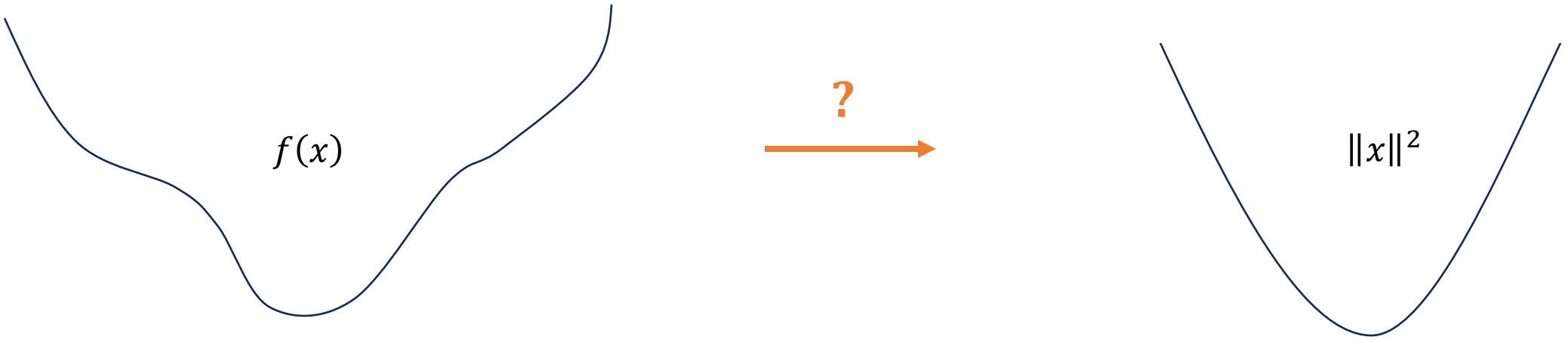

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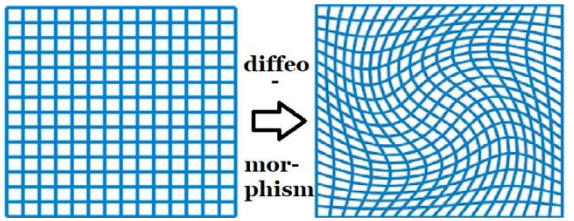

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Cor: There is a complete flat Riemannian metric on $\mathbb{R}^n$ under which f is geodesically convex.

(compare with Rapcsák & Csendes '93)

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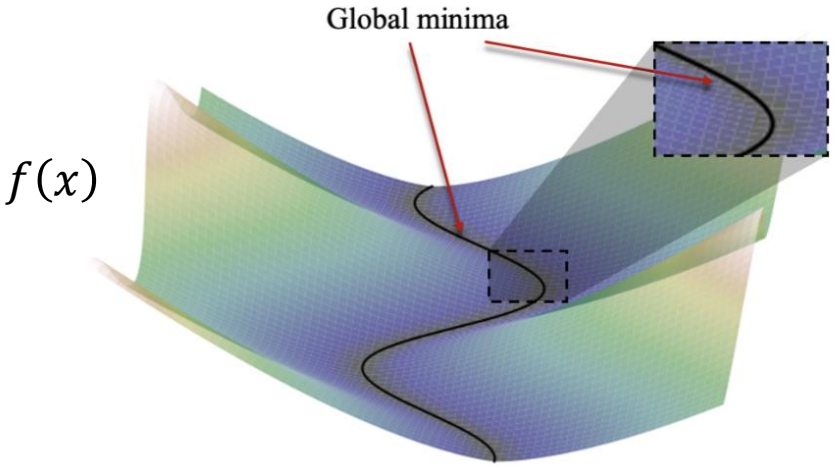

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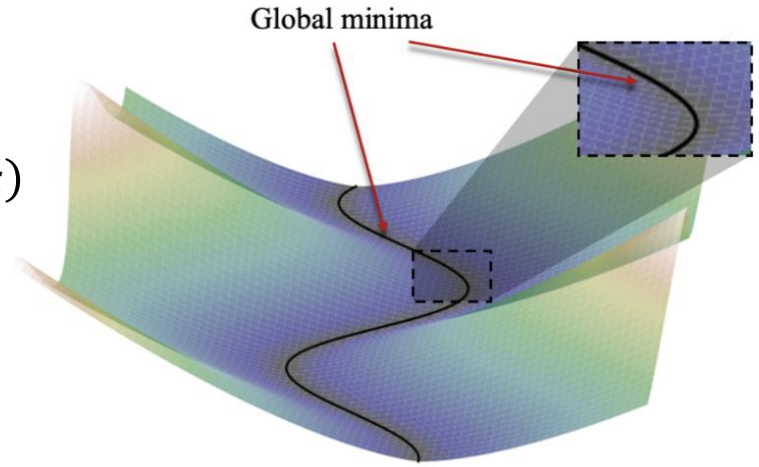
$f(x)$



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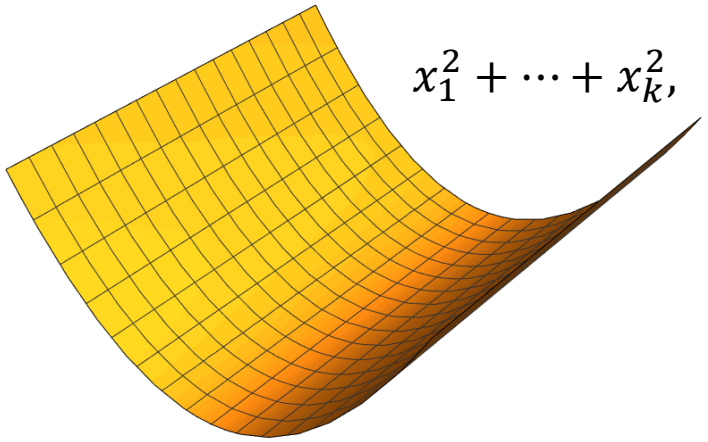

$x_1^2 + \cdots + x_k^2, \qquad k < n$

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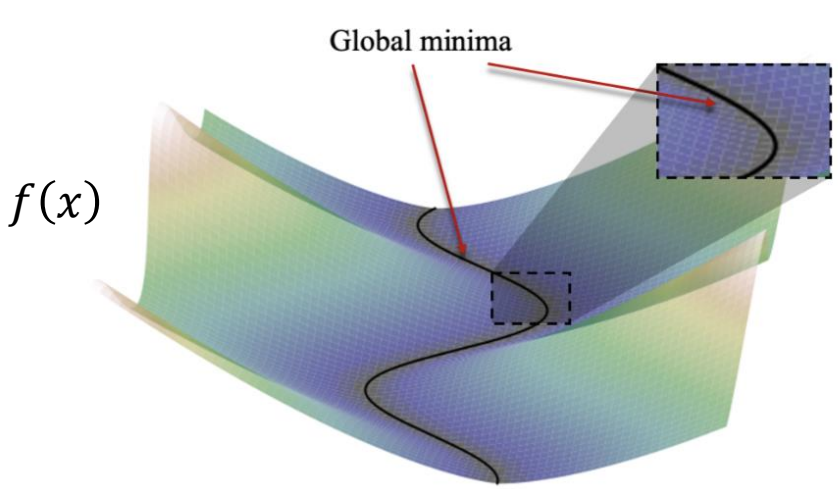

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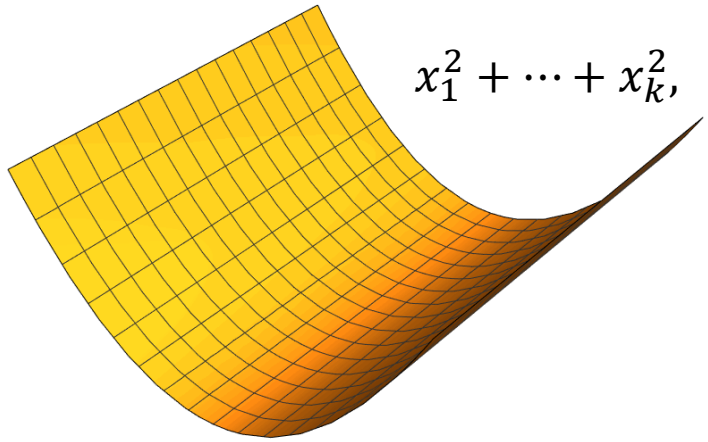

If f is smooth and globally PŁ ~~with unique minimizer,~~ does there exist a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

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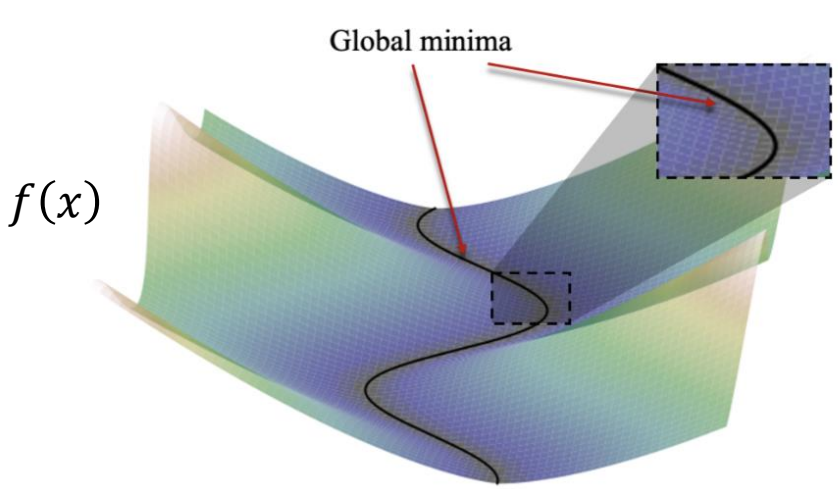

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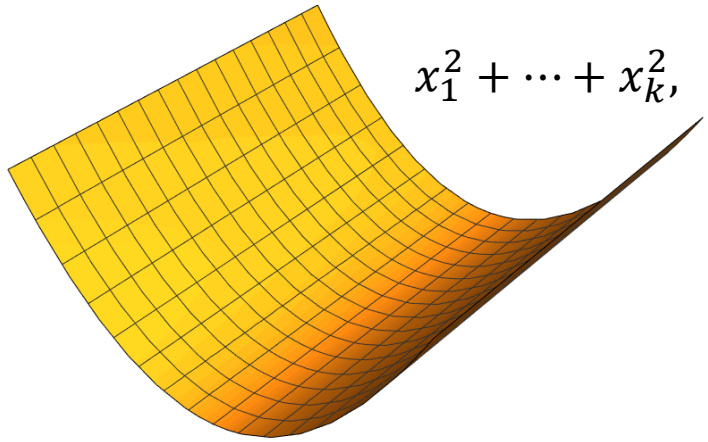

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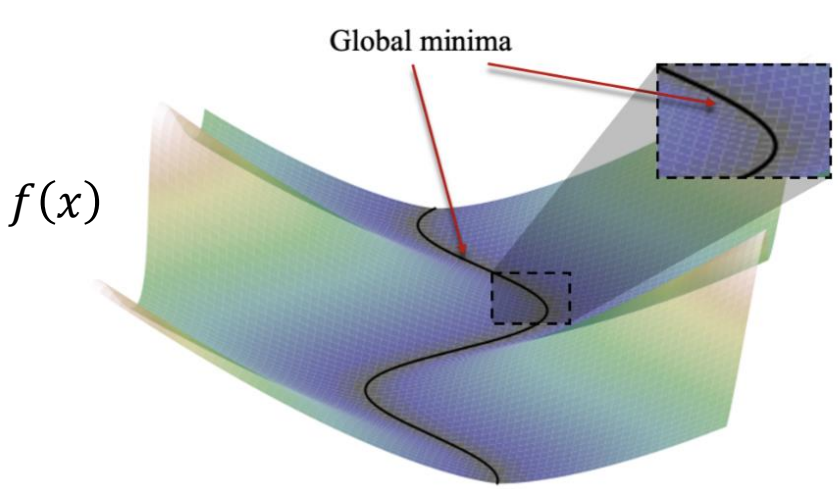

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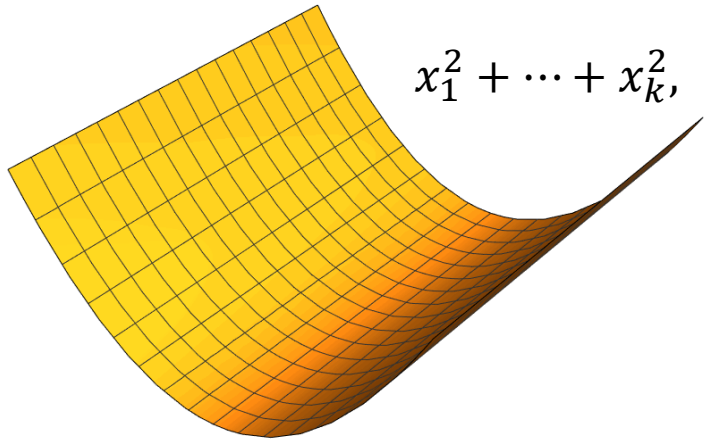

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Short answer: No in general. Yes if $\dim S \leq 2$.

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Main Thm: If f is smooth and globally PŁ, then there exists a diffeomorphism $\phi = (\phi_1, \phi_2) : \mathbb{R}^n \to S \times \mathbb{R}^k$ such that

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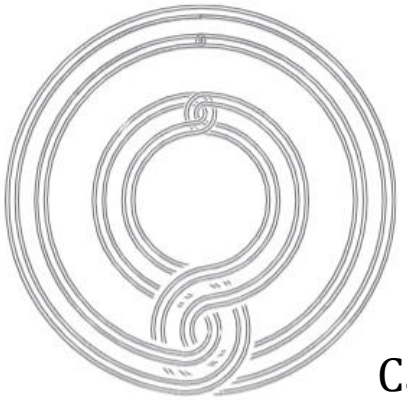
Topological obstruction:

There exist contractible manifolds S of $\dim S \geq 3$ which are not diffeomorphic to $\mathbb{R}^{\dim S}$, and yet $S \times \mathbb{R} \cong \mathbb{R}^{1+\dim S}$

Famous Whitehead manifold ('34)



Calegari '19

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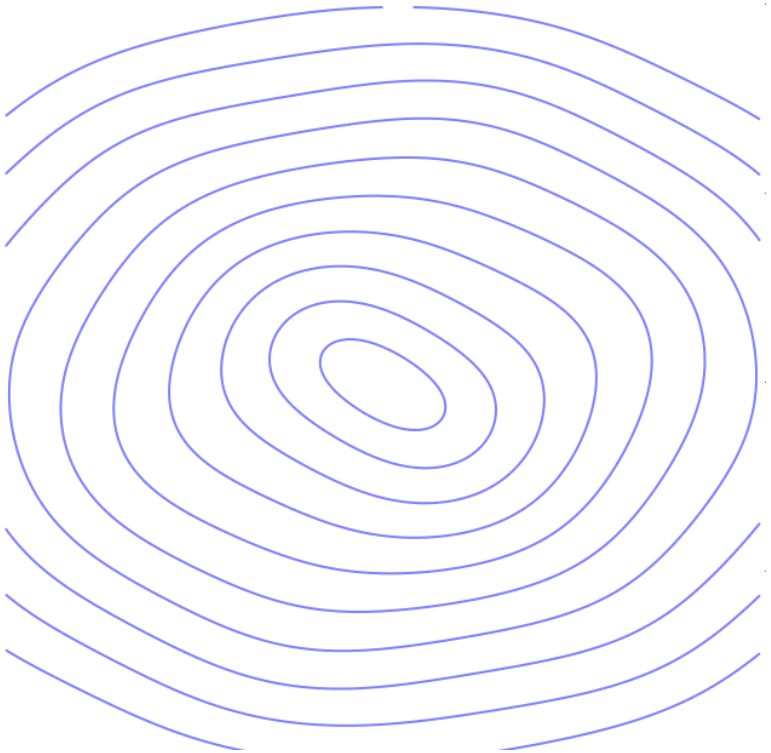

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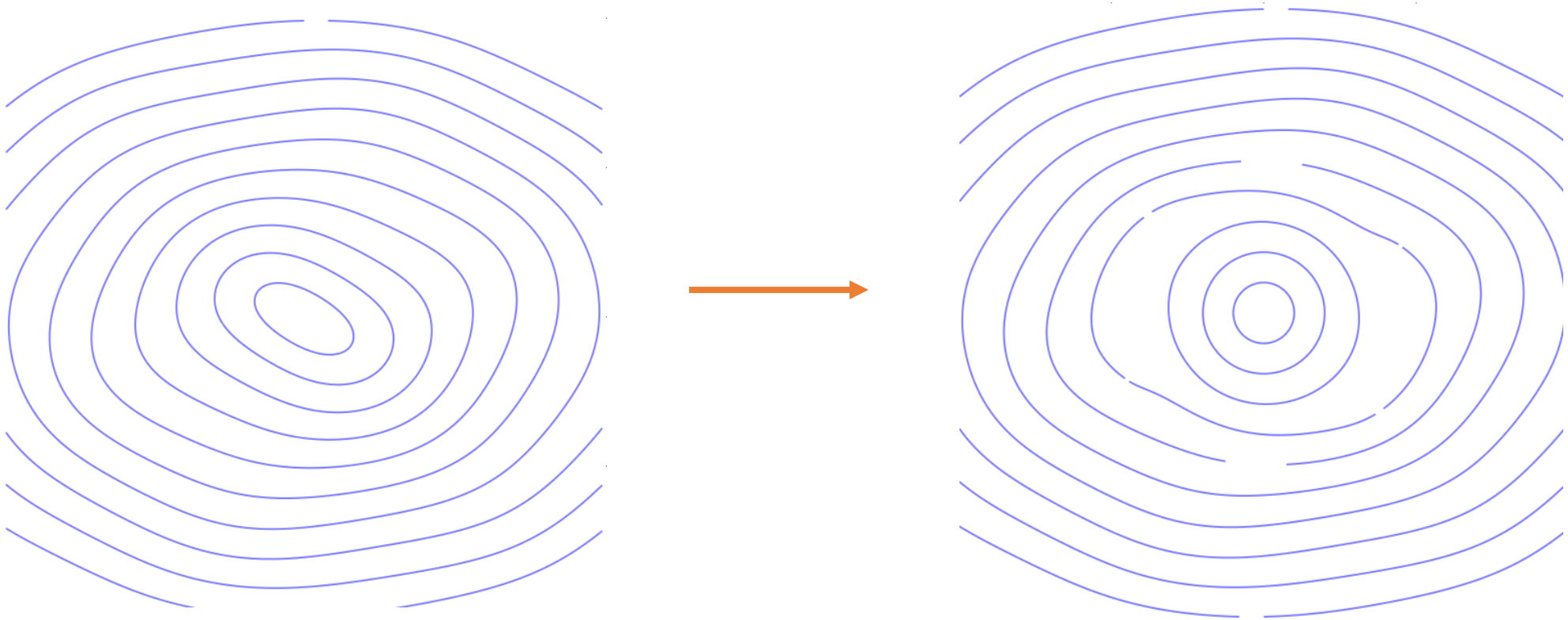

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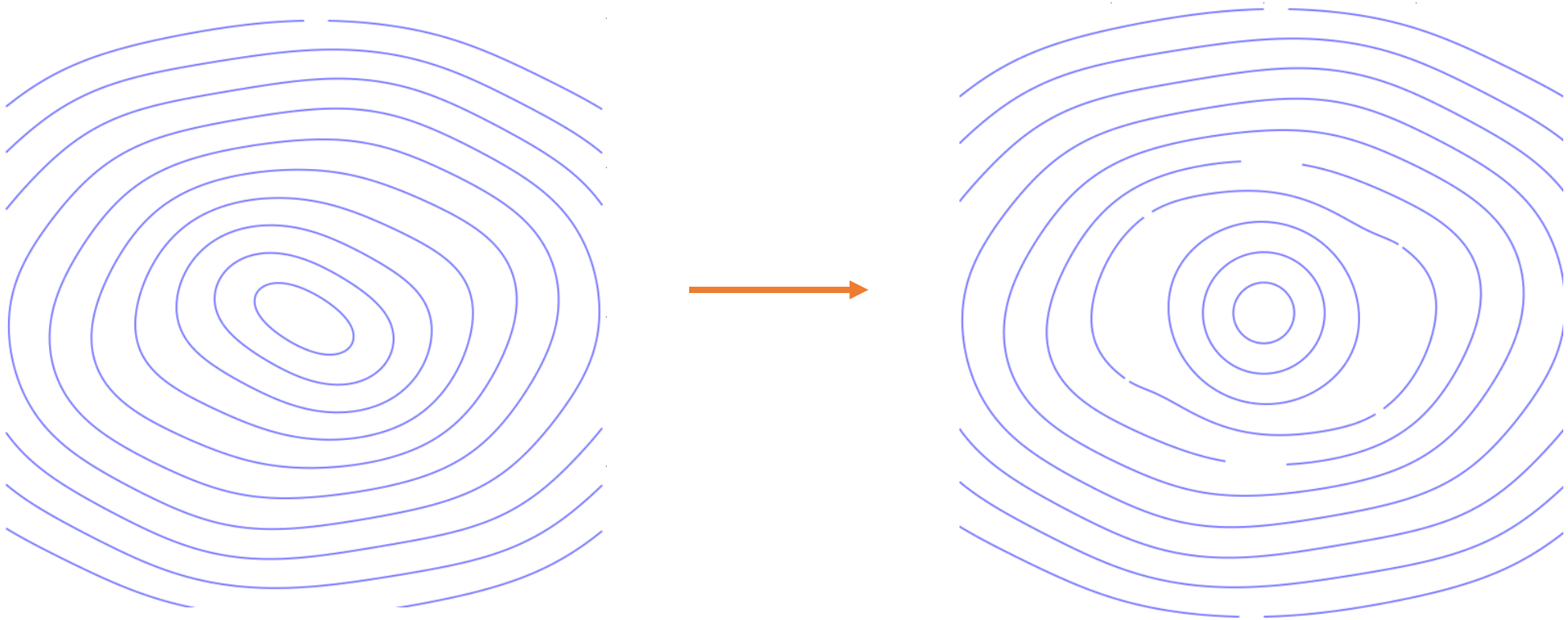

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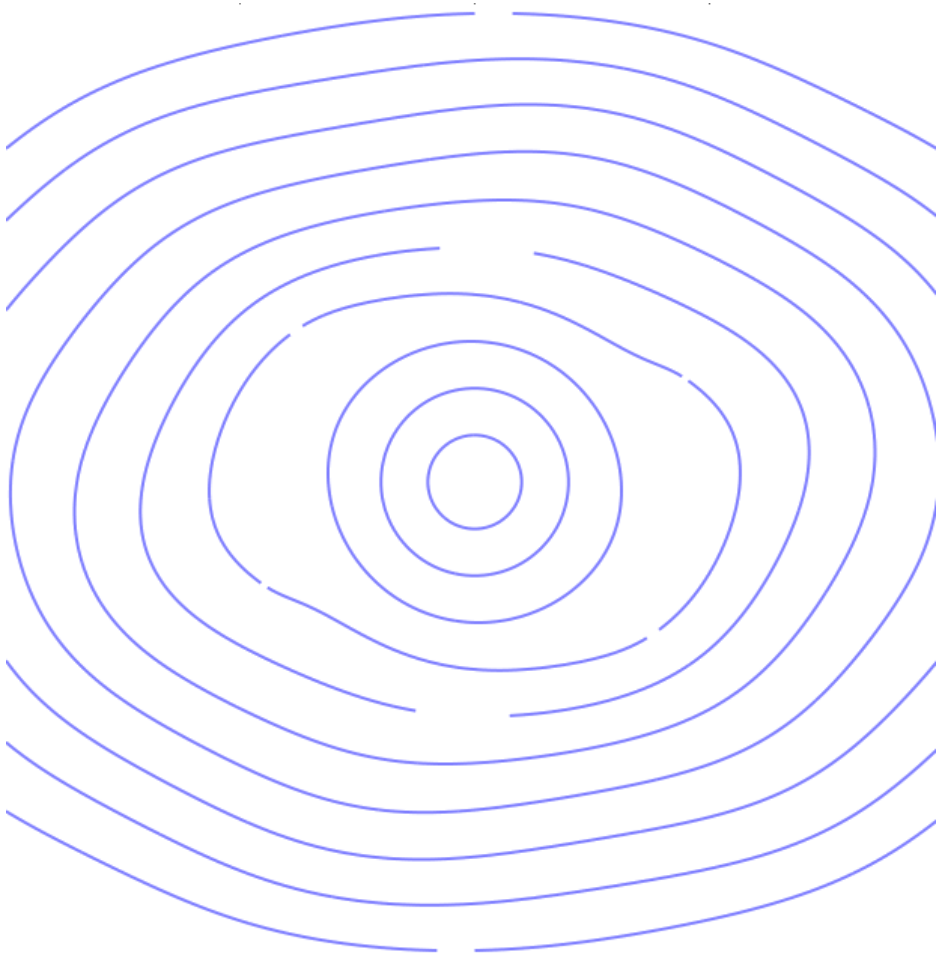

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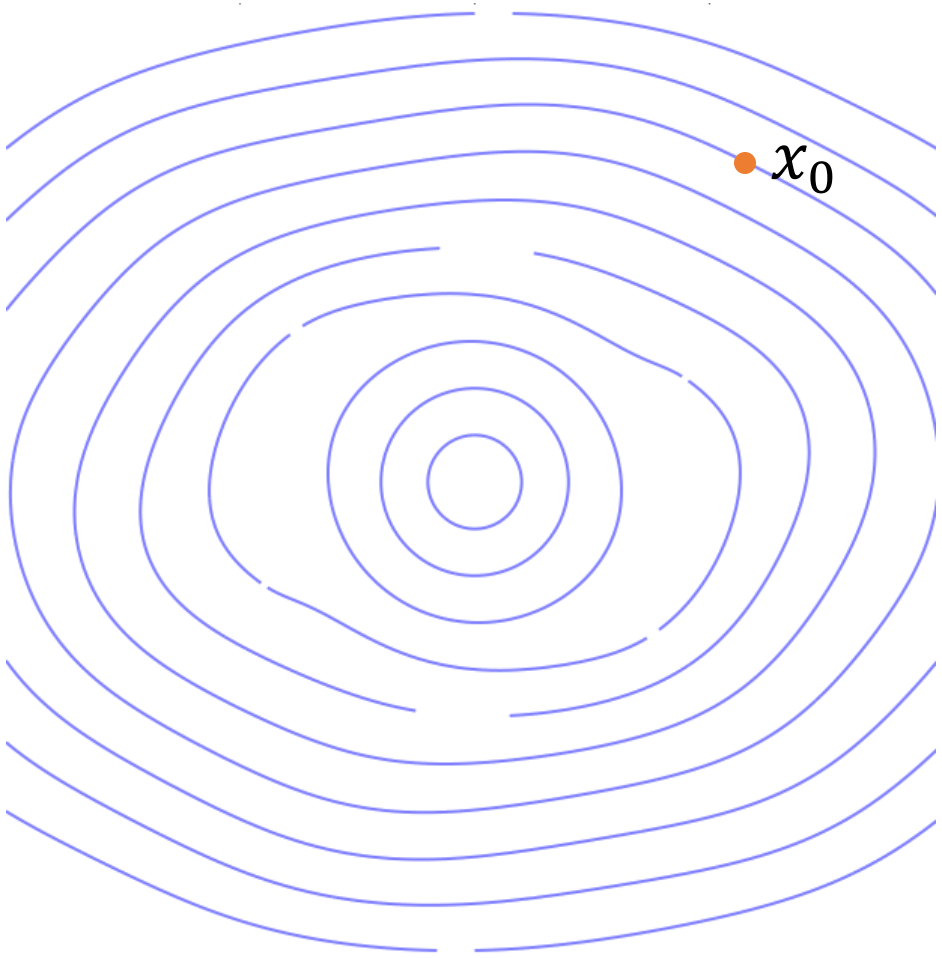

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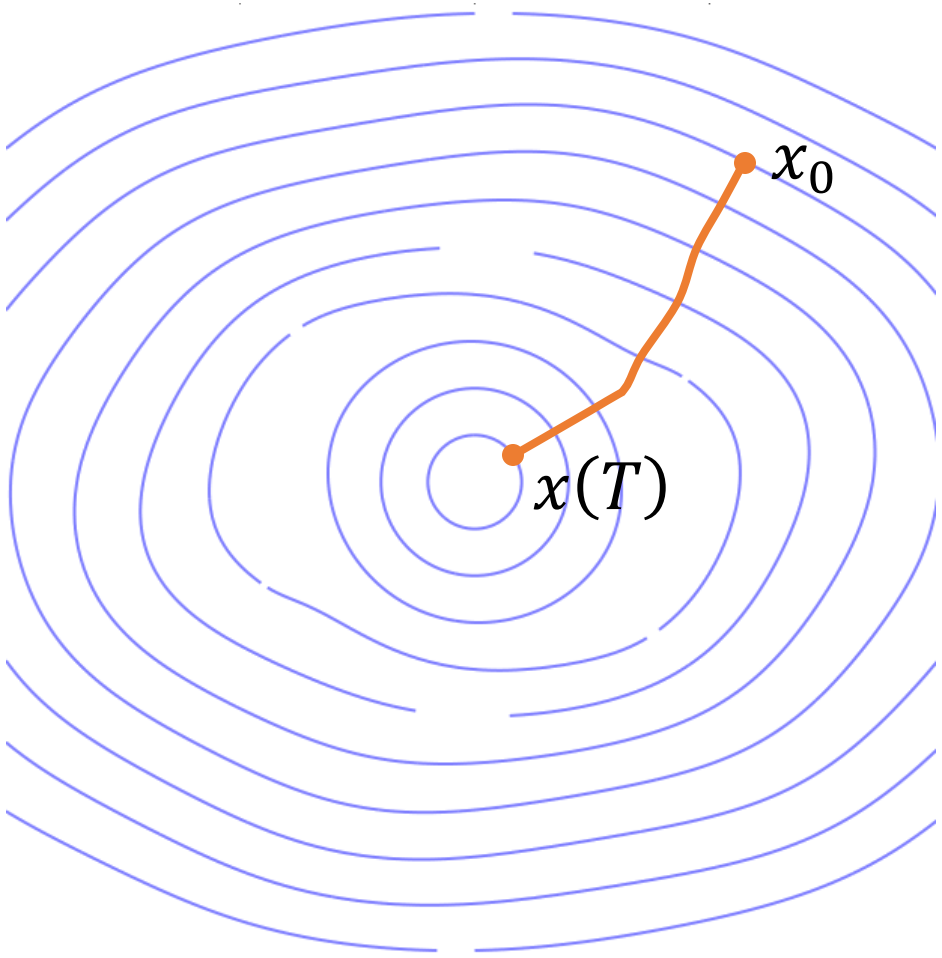

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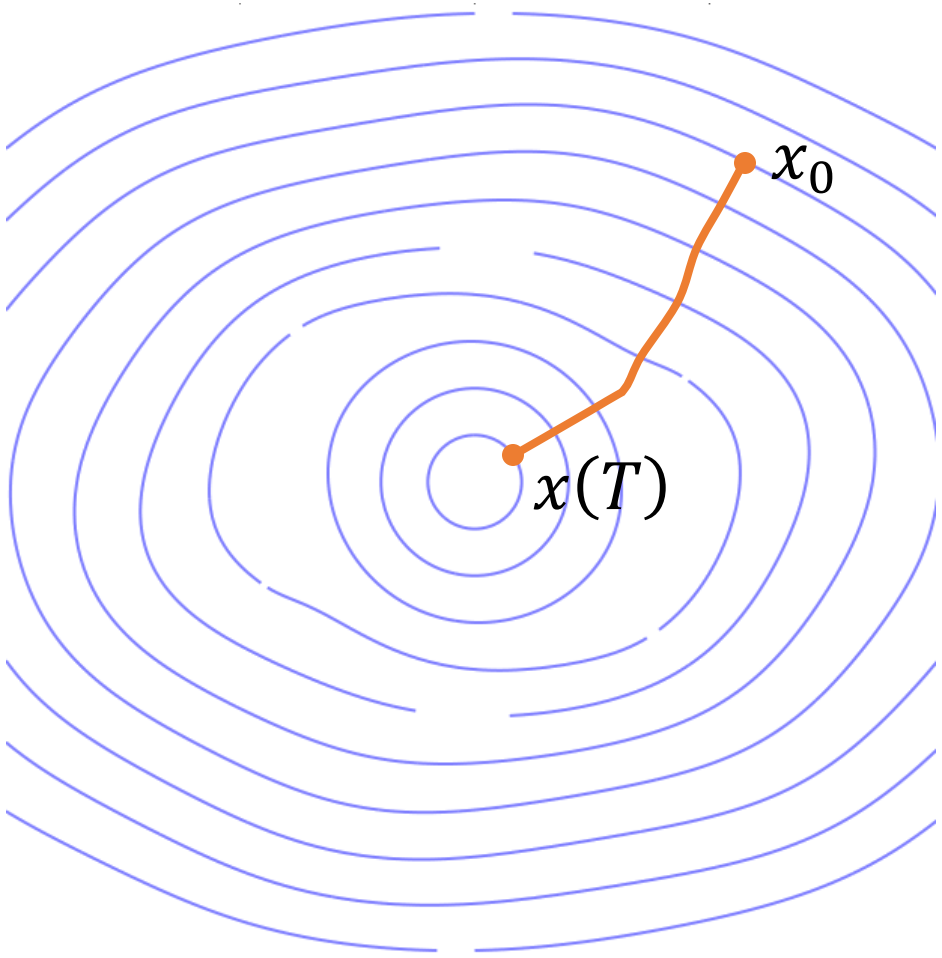

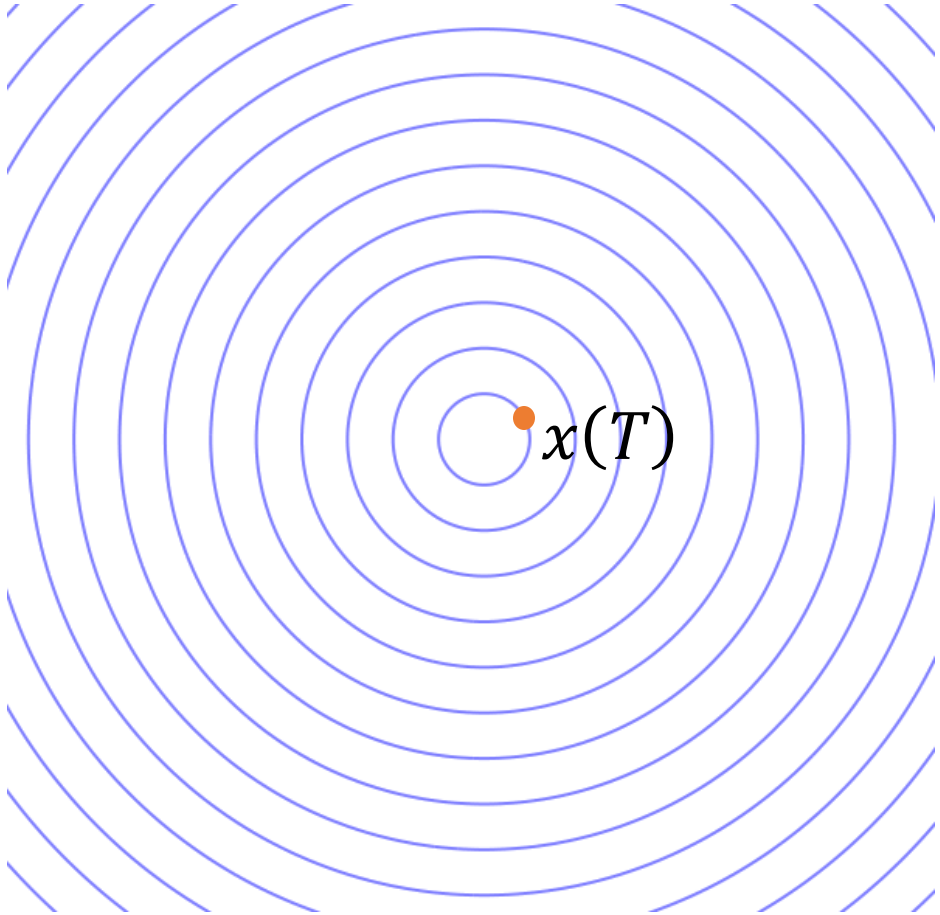

Unique minimizer

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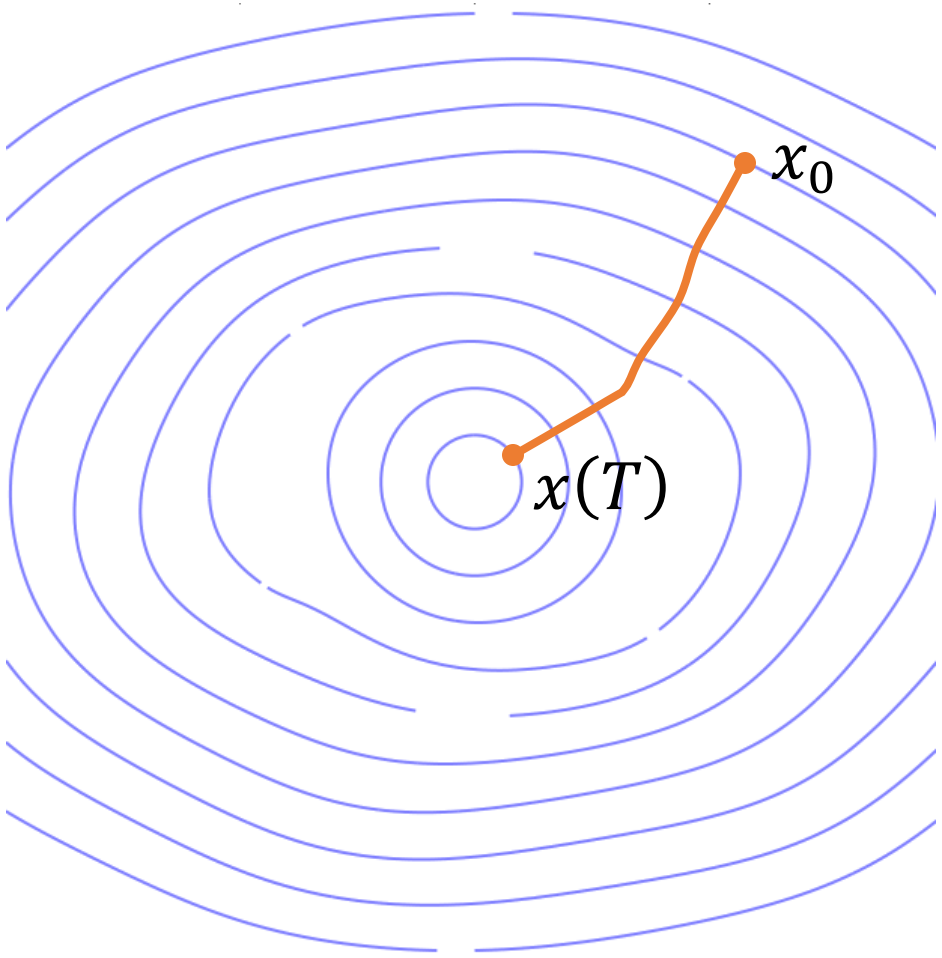

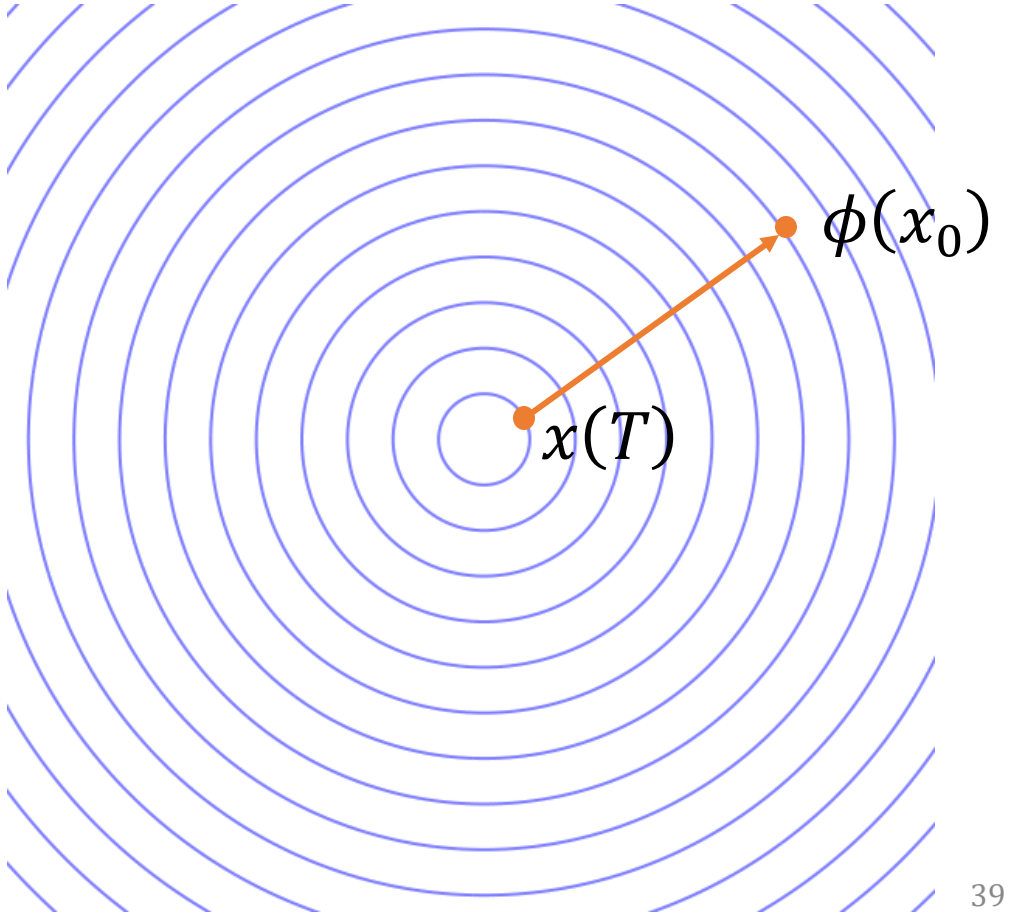

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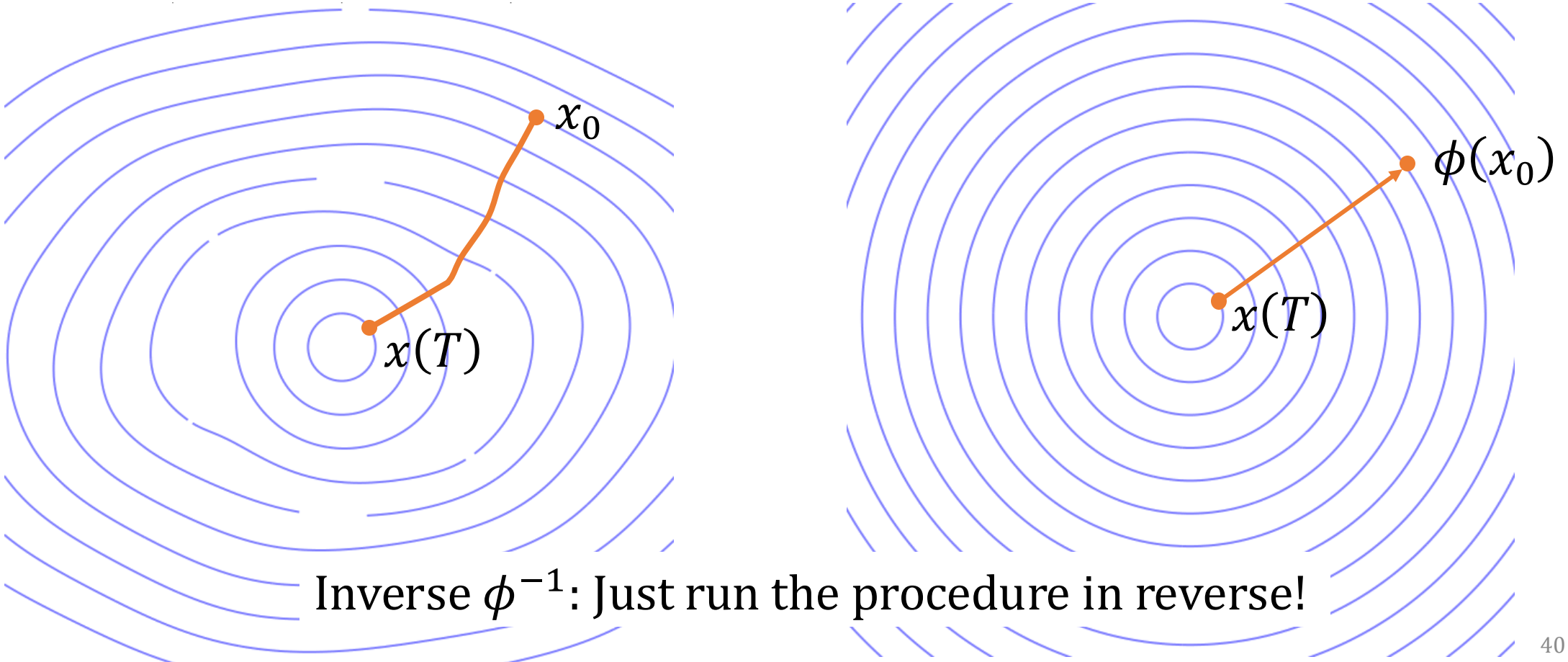

Inverse ϕ^{-1}: Just run the procedure in reverse!

Manifold of minimizers

Main Thm: If f is smooth and globally PŁ, then there exists a diffeomorphism $\phi = (\phi_1, \phi_2) : \mathbb{R}^n \to S \times \mathbb{R}^k$ such that

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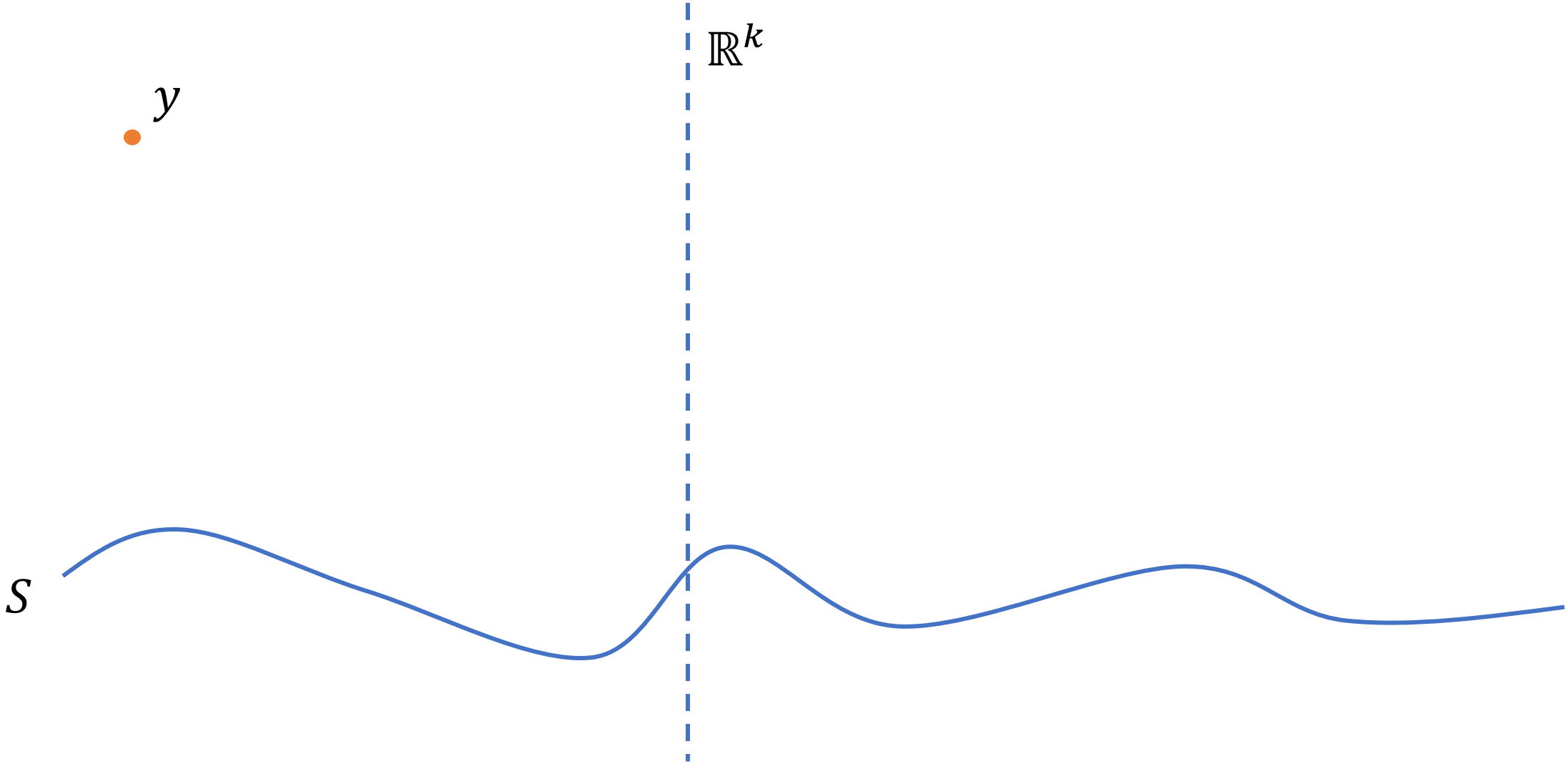

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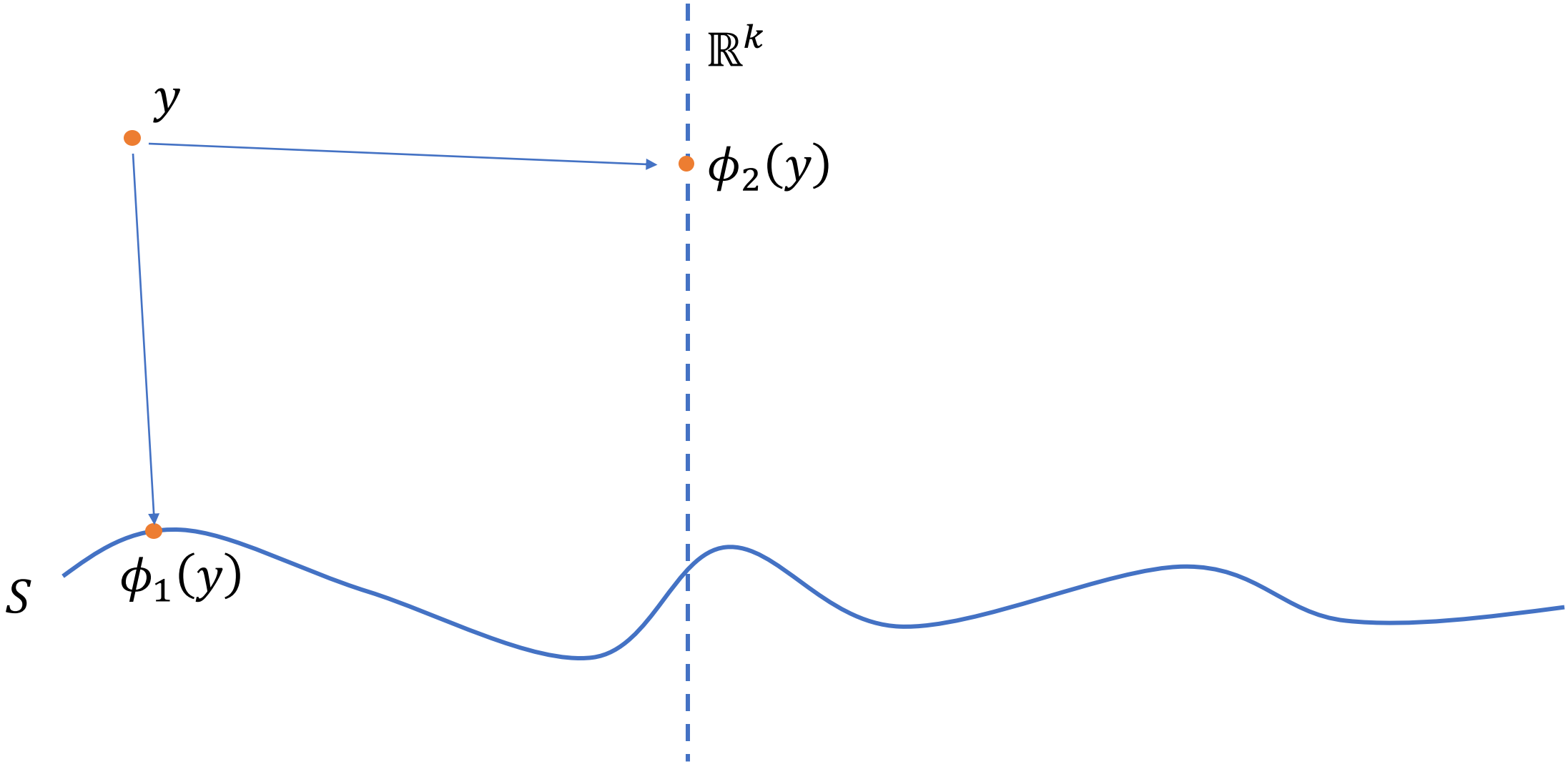

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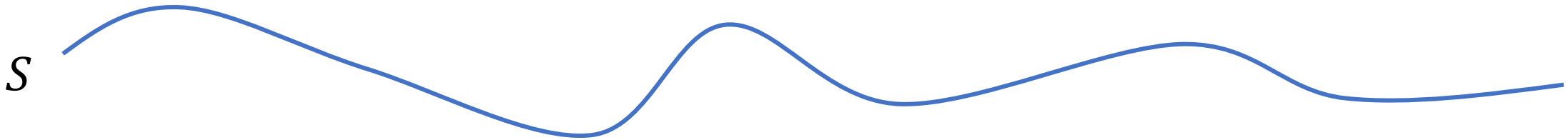

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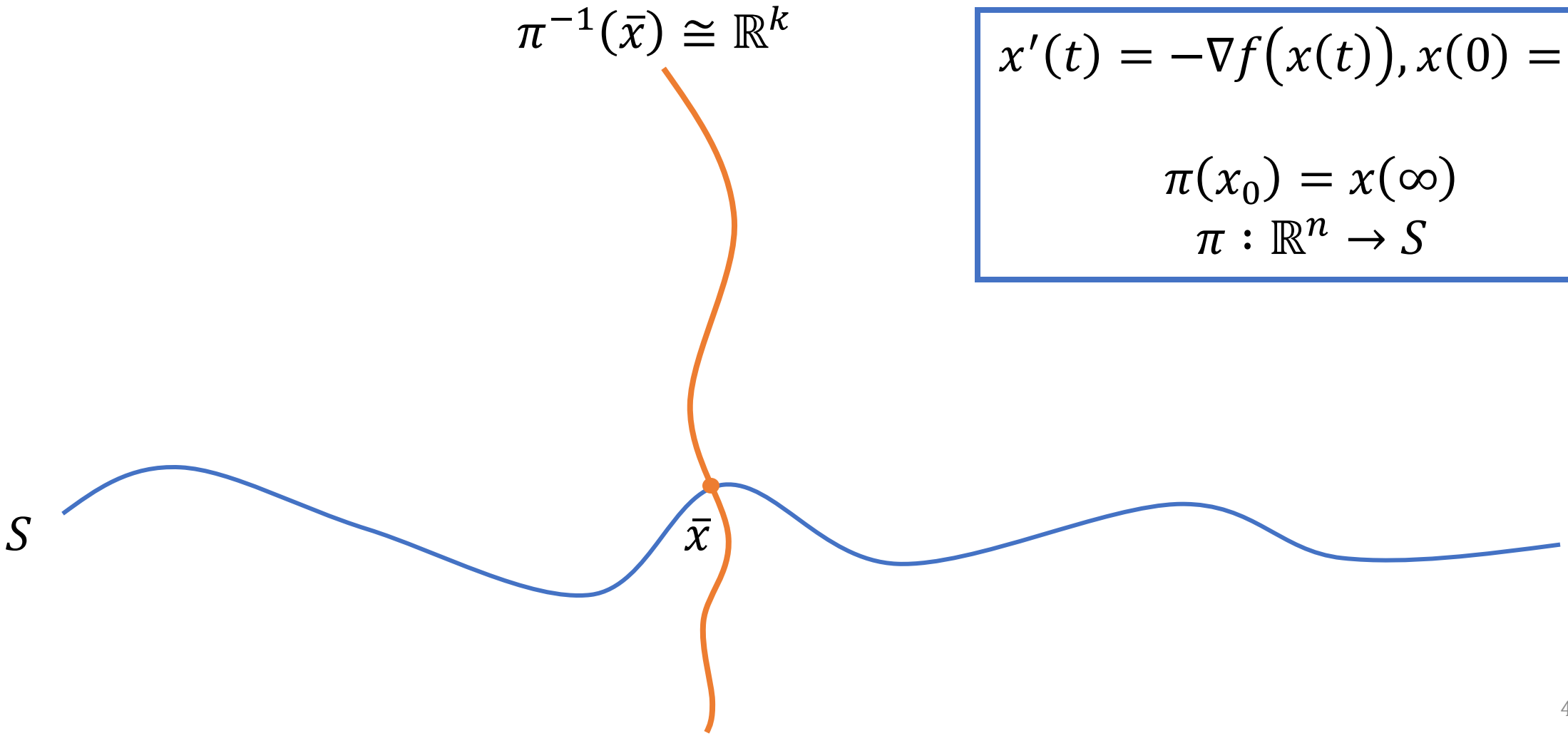

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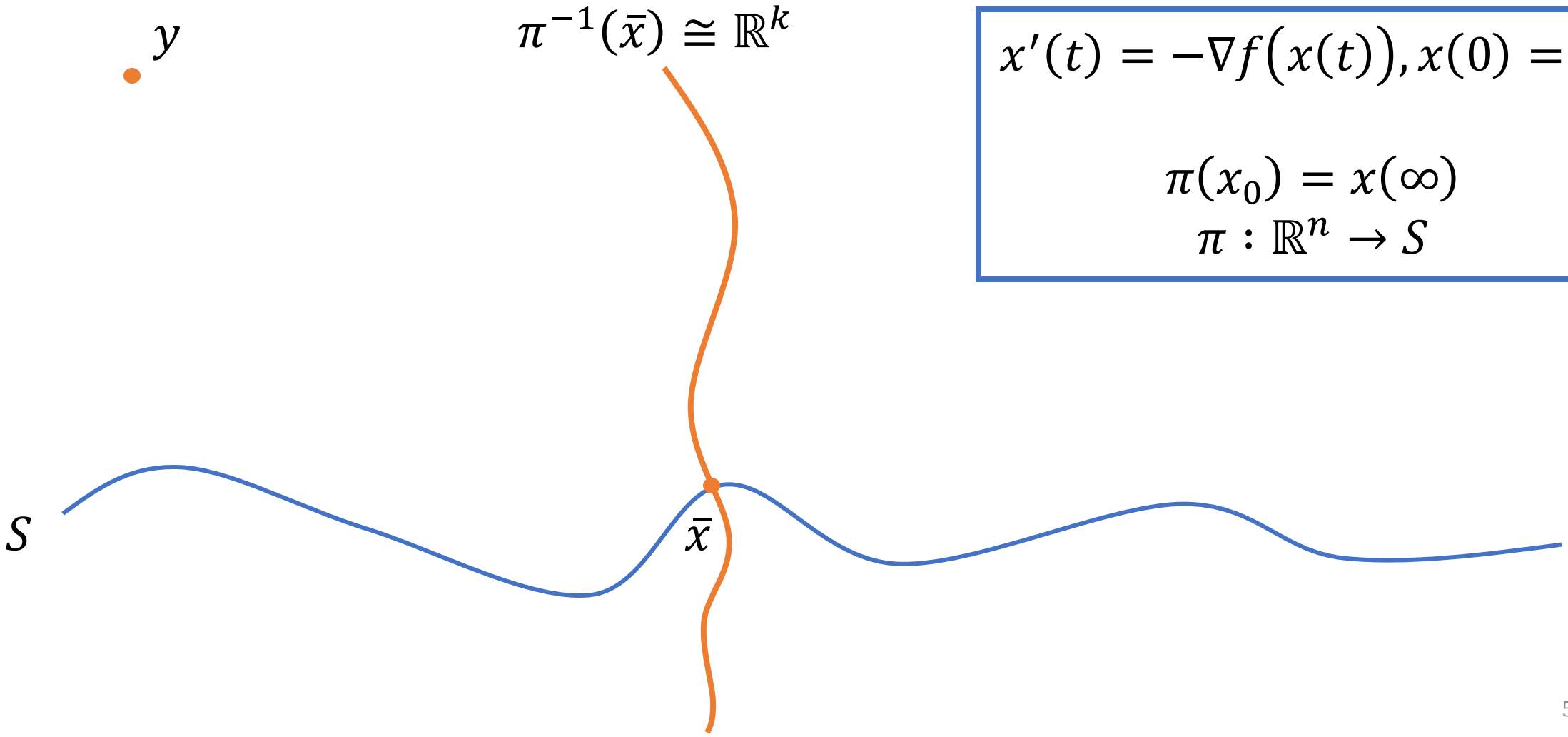

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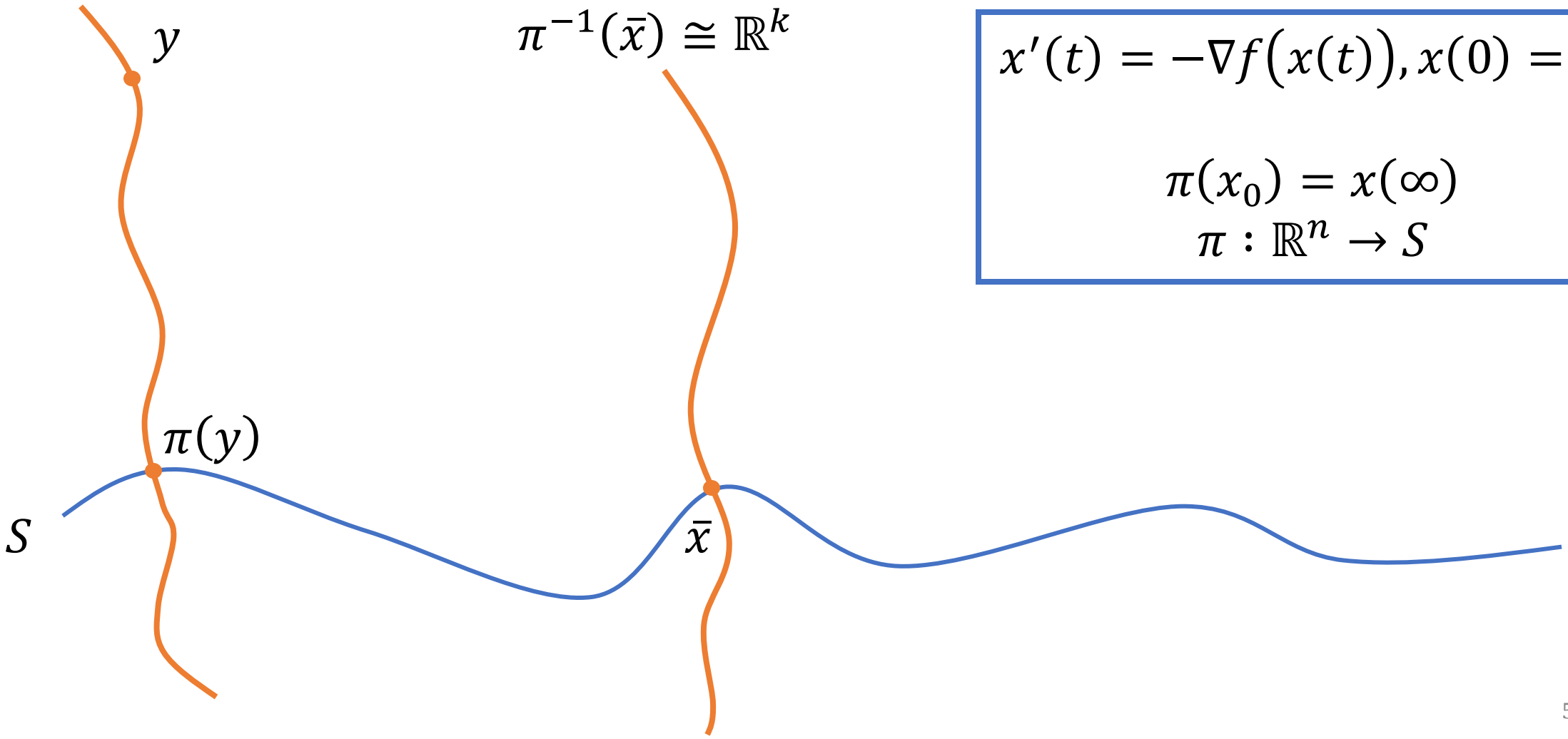

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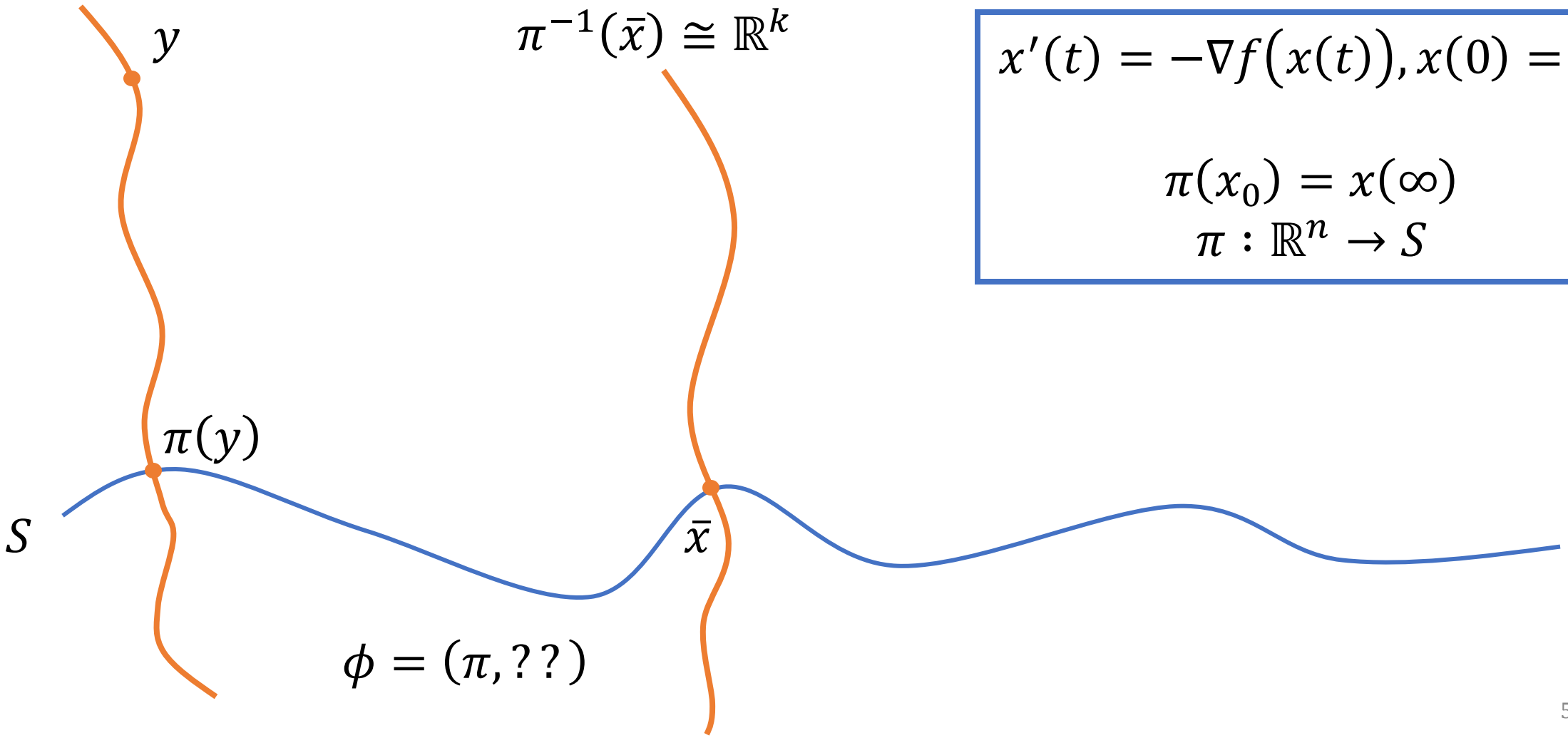

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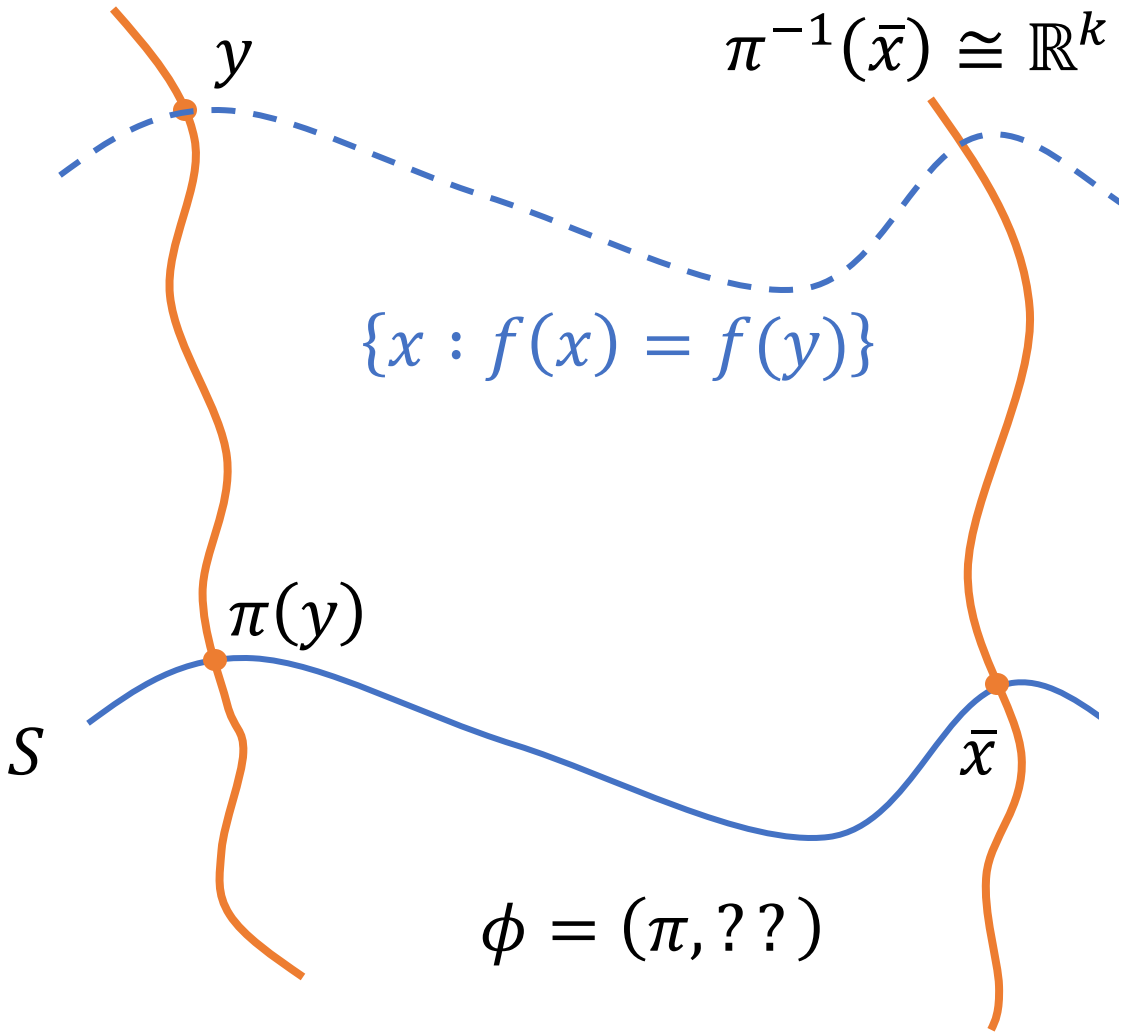

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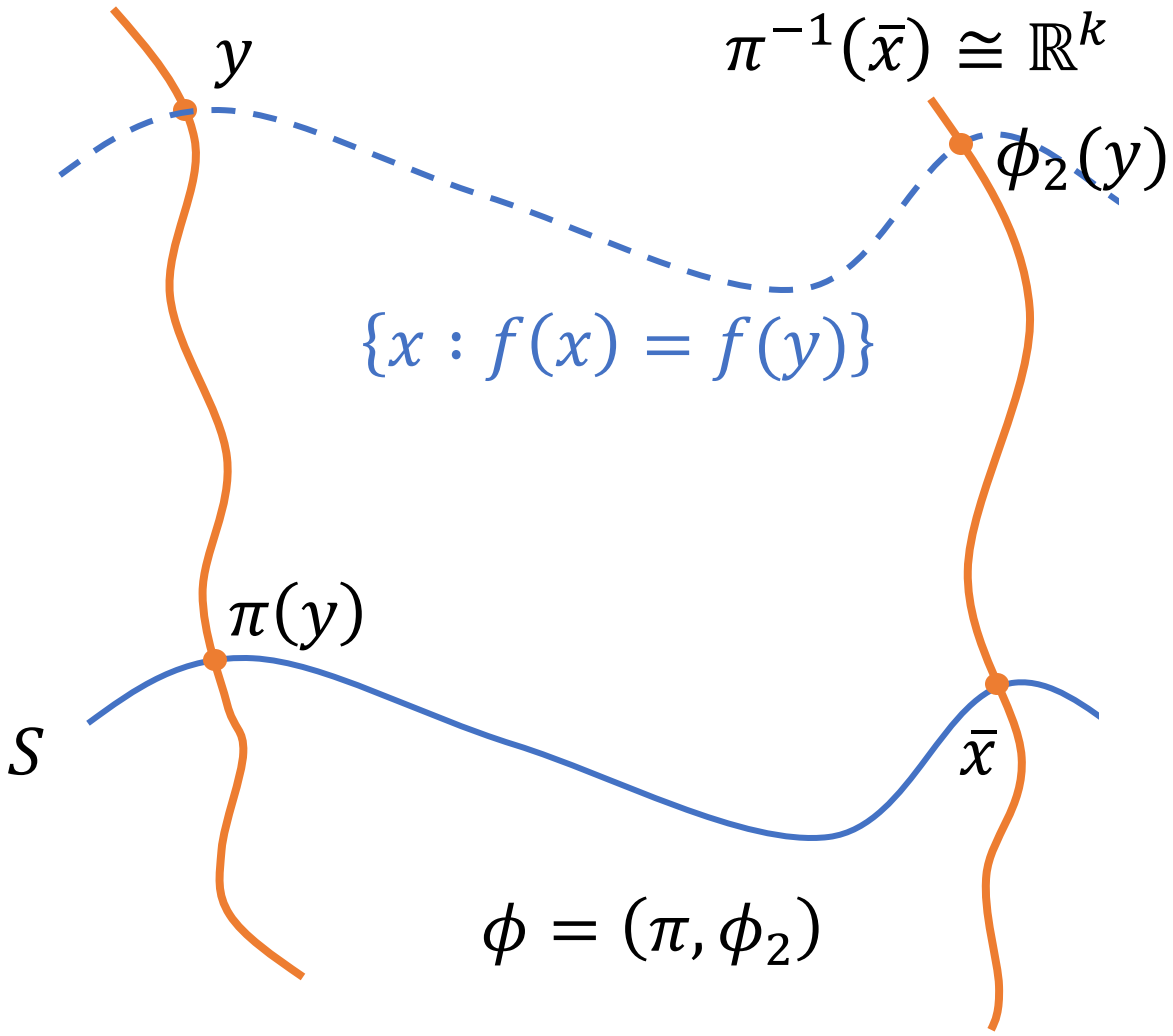

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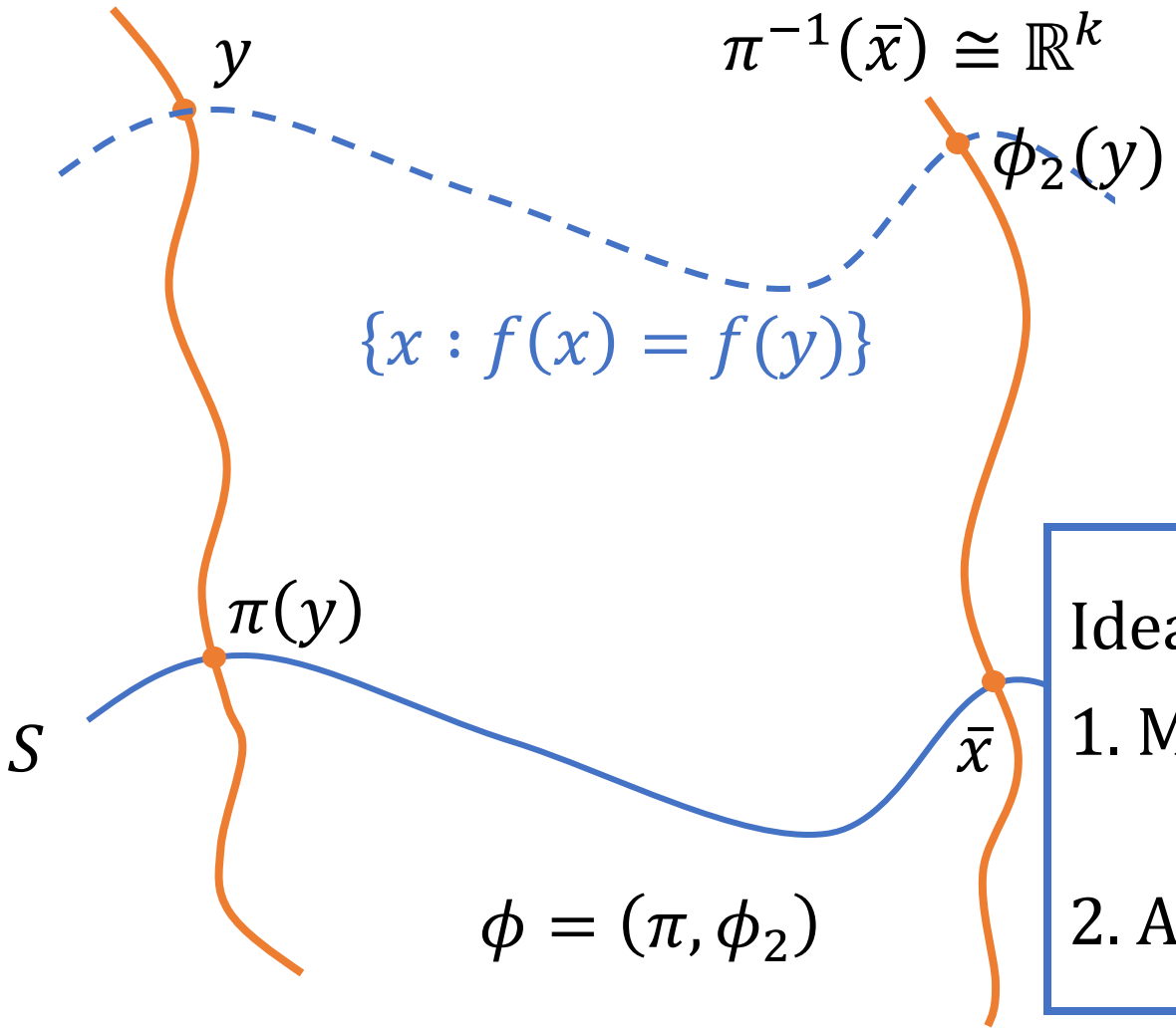

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Idea:

1. Map $\phi_2 : \pi^{-1}\big(\pi(y)\big) \to \pi^{-1}(\bar{x})$, so that
$$f(y) = f\big(\phi_2(y)\big)$$
2. Apply baby thm to f restricted to $\pi^{-1}(\bar{x})$

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Smooth PŁ functions are (almost) always diffeomorphic to squared distances

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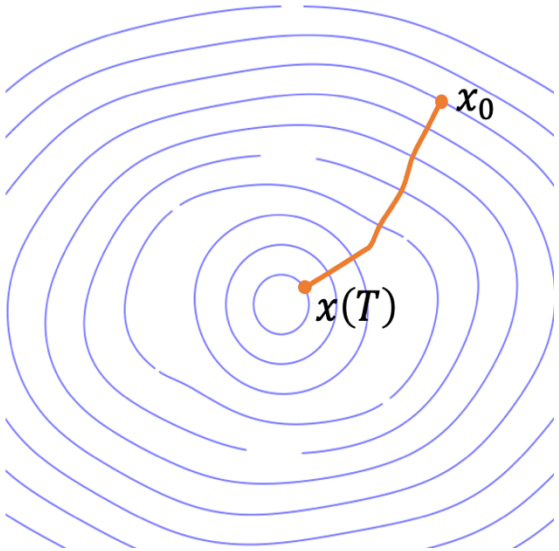
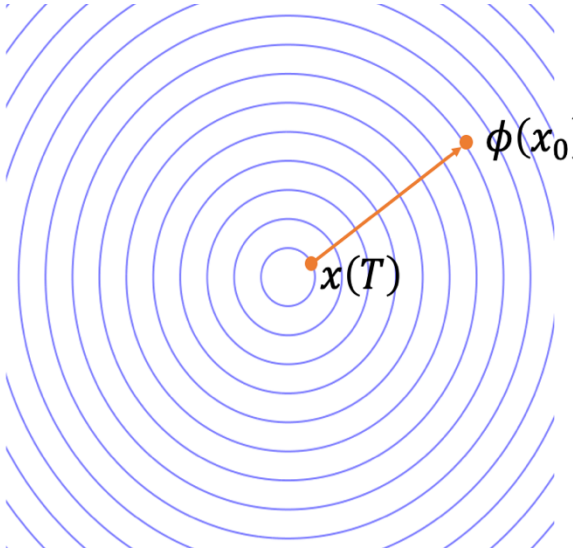
Questions:

PŁ on non-contractible smooth manifolds?

Make results quantitative?





Appendix

Topological obstructions

Cor: If S is simply connected at infinity ($\dim S \geq 5$) OR $\dim S \leq 2$, then there exists a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

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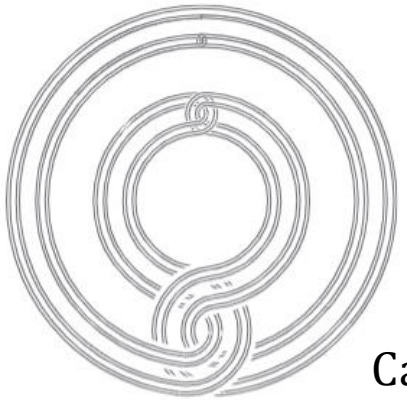

Calegari '19